

ROTOR-VALUED ORIENTATION AS GENERALIZED CLIFFORD GEOMETRY

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ABSTRACT. Grassmann's exterior orientation, encoded by the antisymmetry of the wedge product and inherited in Clifford algebra through the anticommutation of orthogonal generators, is shown to be an order-two rotor acting in the oriented plane. Clifford's geometric algebra therefore appears as the binary limit of an orientation geometry in which reversion carries rotor phase rather than discrete sign reversal. The exchange symmetry induced by reversion admits a harmonic decomposition, and Clifford's geometric product is shown to be the $\mathbb{Z}_2$ Fourier decomposition of transposition. At the root lie the generalized geometric product and phase-valued orientation invariants generalizing signed area and signed volume. I use these developments to derive Morris's generalized Clifford algebra (GCA) canonically from Clifford geometry, identifying the geometric origin of its axiomatic deformation.

1. INTRODUCTION

Generalized Clifford (GCA), Weyl-Heisenberg, and q -deformed algebras appear throughout mathematical physics, quantum theory, representation theory, and noncommutative geometry where the common feature is the replacement of commutation or anticommutation by generalized exchange relations carrying finite phase [5]. Although algebraically well-developed, their underlying geometric meaning has remained comparatively opaque. In particular, deformation parameters q and ω are usually treated as external scalar phases rather than geometric orientation objects. At the same time, exterior and Clifford geometry are fundamentally theories of orientation. In exterior algebra, vectors a, b determine an oriented area element $a \wedge b$, whose magnitude measures parallelogram area while its sign records orientation. The skew-symmetry of the wedge product $a \wedge b = -b \wedge a$ encodes the fact that reversing the ordering of vectors reverses orientation. In geometric algebra this appears through the decomposition $ab = a \cdot b + a \wedge b$, where the wedge product carries the antisymmetric orientation structure of the geometric product (Clifford [2], Hestenes [4]).

Exterior geometry carries a fundamentally binary, or $\mathbb{Z}_2$ -graded, orientation theory, and a central observation of the present work is that this binary sign hides a rotational interpretation—orientation reversal is a half-turn in the oriented bivector plane determined by the vectors themselves. Signed orientation is not merely a combinatorial parity rule, but the simplest instance of a deeper geometry of orientation transport, suggesting a natural generalization in which orientation exchange carries rotor phase rather than collapsing to binary sign. The fundamental mechanism is reversion itself. Exchange is generated by the elementary transposition τ , which reverses the ordering of vectors. In orthogonal Clifford geometry this reversal induces the binary rotor $R = -1$, and more generally generates a local *rotor-valued exchange*

$$ab = (ba)R,$$

with R determined by the geometry of the transposed pair. Exterior forms, determinants, and oriented volume elements also admit rotor-valued refinements in which permutation parity evolves into phase-valued orientation data.

2. THE GEOMETRIC PRODUCT

For unit vectors a, b separated by angle θ , it is well known that the geometric product is decomposed as

$$ab = a \cdot b + a \wedge b = \cos \theta + B \sin \theta,$$

where B is the unit bivector of the oriented ab -plane. Since $B^2 = -1$, one has the rotor exponential identity $e^{\theta B} = \cos \theta + B \sin \theta$. Thus for unit vectors,

$$(1) \quad ab = e^{\theta B}.$$

The geometric product itself defines a rotor in the local wedge plane. Likewise, $ba = e^{-\theta B}$, and reversing order is reversing the product rotor's orientation, or changing the sign of the bivector generator. In particular, for an orthonormal pair $a = e_1, b = e_2$, one has

$$(2) \quad e_1 e_2 = e_{12} = \cos \frac{\pi}{2} + e_{12} \sin \frac{\pi}{2} = e^{\frac{\pi}{2} e_{12}}$$

The geometric product of orthogonal basis vectors is itself a quarter-turn rotor generated by the local bivector plane.

More generally, $ab = a \cdot b + a \wedge b$, and $ba = a \cdot b - a \wedge b$, so the failure of commutativity appears to be entirely controlled by the oriented bivector component. Whenever ba is invertible and non-null, one may write

$$ab = (ba)R, \quad R = (ba)^{-1}ab \\ = a^{-1}b^{-1}ab.$$

In the unit vector case $a^2 = b^2 = 1$, this simplifies to $R = abab = (ab)^2$. Since $ab = e^{\theta B}$, one obtains $R = e^{2\theta B}$, and,

$$(3) \quad ab = (ba)e^{2\theta B}.$$

So we note, reversing the order of vectors gives rise to a local right exchange rotor R generated by the same bivector plane as the geometric product itself.

2.1. Metric-Orientation Coupling. In Grassmann's exterior algebra [3] one has $a \wedge b = -b \wedge a$, so exchanging vectors reverses orientation by a sign. However, the sign reversal is universal and independent of the metric magnitude of the bivector itself. This is geometrically significant: the wedge $a \wedge b$ simultaneously encodes both area magnitude and orientation plane, yet exterior antisymmetry acts only on the orientation component.

Indeed, in Euclidean signature one has $(a \wedge b)^2 = -|a \wedge b|^2$, so the bivector $a \wedge b$ contains both metric magnitude and orientation plane information. Writing $a \wedge b = |a \wedge b| B_{ab}$, exterior antisymmetry acts only on the orientation factor B_{ab} , leaving the metric magnitude $|a \wedge b|$ unchanged. If $B_{ab} = \frac{a \wedge b}{|a \wedge b|}$ denotes the unit bivector of the oriented ab -plane, then $B_{ab}^2 = -1$, and hence $-1 = e^{\pi B_{ab}}$. In the orthonormal case, one has $|a \wedge b| = 1$, and therefore $B_{ab} = a \wedge b$, so the sign reversal may be interpreted geometrically as

$$-(a \wedge b) = (a \wedge b)e^{\pi(a \wedge b)}.$$

More generally, exterior antisymmetry becomes $a \wedge b = (b \wedge a)e^{\pi B_{ab}}$. Signed orientation is a half-turn rotor generated by the local oriented plane. The outer product induces an order-two rotor geometry latent within exterior algebra.

The exchange relation also shows that a rotor-valued exchange cannot be treated as a deformation of the exterior part alone. Starting from the ordinary Clifford decomposition, suppose that the reversed ordering differs from the original product by a right exchange rotor, $ab = (ba)R$. Equivalently, $ba = (ab)R^{-1}$. We find,

$$ab = (ba)R. \\ = (a \cdot b - a \wedge b)R \\ = (a \cdot b)R - (a \wedge b)R.$$

Moving all terms to one side and factoring on the right gives

$$(a \cdot b)(1 - R) + (a \wedge b)(1 + R) = 0.$$

Thus the scalar metric component and the bivector orientation component are not independent under a nontrivial rotor-valued exchange. If a and b are parallel, then $a \wedge b = 0$, and the condition is $(a \cdot b)(1 - R) = 0$. For nonzero $a \cdot b$, this forces $R = 1$, so parallel vectors commute. If a and b are orthogonal, then $a \cdot b = 0$, and the condition is $(a \wedge b)(1 + R) = 0$. For nonzero $a \wedge b$, this forces $R = -1$. For a generic pair, both $a \cdot b$ and $a \wedge b$ are nonzero, and the exchange relation therefore implies

$$(4) \quad (a \cdot b)(1 - R) = -(a \wedge b)(1 + R),$$

so the metric and orientation sectors are phase-coupled by the exchange rotor R . In orthogonal Clifford algebra this coupling collapses to the binary cases $R = \pm 1$, giving the familiar separation between symmetric metric part and antisymmetric exterior part.

2.2. Character Decomposition of the Geometric Product.

Theorem 2.1 (The Geometric Product as a Fourier Decomposition). *Clifford's geometric product is the $\mathbb{Z}_2$ Fourier decomposition of the transposition operator*

$$\tau(ab) = ba.$$

Proof. Let τ denote the elementary reversion operator on an ordered vector product, $\tau(ab) = (ba)$. In the orthogonal Clifford basis, exchange gives the binary exchange rotor $R = -1$, so $\tau(ab) = (ba) = (ab)R^{-1} = -ab$. To determine the exchange sectors of the geometric product, one seeks a basis in which the exchange operator is diagonal, i.e. a basis of eigenvectors of τ . More specifically, $\tau \in S_n$ is a transposition, and since exchanging twice returns the original order, τ is an involution, and its possible eigenvalues are $\lambda = \pm 1$. The eigenspaces are obtained by forming the symmetric and antisymmetric combinations of ab . Indeed, $\tau(ab + ba) = ba + ab = ab + ba$, so

$ab + ba$ lies in the $+1$ -eigenspace of τ , while $\tau(ab - ba) = ba - ab = -(ab - ba)$, so $ab - ba$ lies in the -1 -eigenspace of τ . Thus the projectors onto the two exchange eigenspaces are

$$\Pi_+ = \frac{1}{2}(1 + \tau), \quad \Pi_- = \frac{1}{2}(1 - \tau).$$

Since $\tau^2 = 1$, the exchange action generates the regular representation of $\mathbb{Z}_2$ on the orbit $\text{span}\{ab, ba\}$. In the ordered basis (ab, ba) , the normalized character table diagonalizes the action of $\rho(\tau)$

$$F_2 \rho(\tau) F_2^{-1} = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Thus the Fourier basis separates the trivial and sign representations of exchange.

$$\Pi_+ = \frac{1}{2}(1 + \tau), \quad \Pi_- = \frac{1}{2}(1 - \tau),$$

are the spectral projectors of τ , or equivalently the character idempotents of the group algebra $\mathbb{C}[\mathbb{Z}_2]$. Applying these to ab gives

$$\begin{aligned} \Pi_+(ab) &= \frac{1}{2}(1 + \tau)(ab) = \frac{1}{2}(ab + \tau(ab)) = \frac{1}{2}(ab + ba) = \frac{1}{2}(ab + (ab)R^{-1}) = \frac{1}{2}ab(1 + R^{-1}), \\ \Pi_-(ab) &= \frac{1}{2}(1 - \tau)(ab) = \frac{1}{2}(ab - \tau(ab)) = \frac{1}{2}(ab - ba) = \frac{1}{2}(ab - (ab)R^{-1}) = \frac{1}{2}ab(1 - R^{-1}). \end{aligned}$$

For vectors, these projectors recover Clifford's symmetric and antisymmetric products. Hence

$$ab = \Pi_+(ab) + \Pi_-(ab) = a \cdot b + a \wedge b.$$

□

The character decomposition above leads to a finite cyclic generalization in which the binary exchange rotor $R = -1$ is replaced by a higher-order exchange rotor. Interestingly, the same finite cyclic exchange may arise from two distinct mechanisms. In orthogonal Clifford geometry, the angle of the geometric product rotor $ab = e^{\theta B_{ab}}$ is induced by the metric orientation spacing between vectors (2). The corresponding exchange rotor is $R = (ab)^2 = e^{2\theta B_{ab}}$, where $\theta = \pi/2$ follows from the orthogonality of the generators (3). If the vectors are instead separated by $\theta = \pi/d$, then $R^d = 1$, yielding a finite cyclic exchange.

In contrast, Weyl systems [7] use matrices to exhibit the same exchange transport through an algebraic mechanism rather than metric spacing. The generators satisfy $UV = \omega VU$, $\omega^d = 1$, [6] where the cyclicity arises from the finite order and noncommutation of the clock U and shift V matrices. Importantly, the Weyl generators remain orthogonal in the Frobenius sense, $\langle U, V \rangle_F = \text{tr}(U^\dagger V) = 0$, independently of d . The additive metric geometry does not detect the cyclic exchange angle, reflecting two different geometric notions. The Frobenius inner product measures additive operator alignment, while the exchange invariant

$$R = (ba)^{-1}(ab) = [b, a]_G$$

measures multiplicative orientation transport through the multiplicative group commutator of the ordered pair.

Definition (Generalized Geometric Product). *Let $\mathcal{V}$ be a vector space equipped with the elementary reversion operation τ , inducing a relation $ab = (ba)R$, where $R = e^{2\theta B}$ is an invertible exchange rotor determined by the pair (a, b) .*

$$\tau : \mathcal{V} \otimes \mathcal{V} \rightarrow \mathcal{V} \otimes \mathcal{V}, \quad \tau^m(ab) = (ab)R^{-m}, \quad m \in \mathbb{Z}.$$

A generalized geometric product is a bilinear product whose multiplication admits a harmonic decomposition under the action of τ .

In the finite cyclic case, when $m \in \mathbb{Z}_d$ and $R^d = 1$, the product decomposes into $\mathbb{Z}_d$ -exchange sectors

$$(5) \quad ab = \sum_{k=0}^{d-1} \Pi_k(ab), \quad \Pi_k = \frac{1}{d} \sum_{m=0}^{d-1} R^{-km} \tau^m.$$

Now, the local exchange operator satisfies $\tau^d = 1$, with eigenvalues $1, R, R^2, \dots, R^{d-1}$. The symmetric/antisymmetric split is then replaced by a $\mathbb{Z}_d$ -exchange decomposition, and these projectors are in fact the character idempotents of the cyclic group algebra $\mathbb{C}[\mathbb{Z}_d]$, with $\Pi_k^2 = \Pi_k$, $\Pi_j \Pi_k = 0$ ($j \neq k$), $\sum_{k=0}^{d-1} \Pi_k = 1$. For $d = 3$, one obtains three sectors $ab = (ab)_0 + (ab)_1 + (ab)_2$, with

$$\begin{aligned} \Pi_0 &= \frac{1}{3}(1 + \tau + \tau^2), & \Pi_1 &= \frac{1}{3}(1 + R^{-1}\tau + R^{-2}\tau^2), & \Pi_2 &= \frac{1}{3}(1 + R^{-2}\tau + R^{-4}\tau^2) \\ & & &= \frac{1}{3}(1 + R^2\tau + R\tau^2), & &= \frac{1}{3}(1 + R\tau + R^2\tau^2). \end{aligned}$$

The sector $(ab)_0$ is the exchange-invariant or metric-like part. The remaining sectors $(ab)_1$ and $(ab)_2$ are conjugate orientation-phase sectors. Thus, for $d = 3$, the analogue of the wedge/outer part no longer consists of a single

antisymmetric component. It splits into two cyclic orientation sectors, corresponding to the two nontrivial cubic roots of unity.

3. CONCLUSIONS

The harmonic exchange decomposition developed here reveals the geometric product as the $\mathbb{Z}_2$ character decomposition of transposition, while generalized Clifford algebras and Weyl-type exchange structures emerge naturally from higher-order exchange rotors. Recognizing $-1 = e^{\pi B}$ as a hidden rotor suggests a systematic theory of geometric algebra with rotor-valued orientation. We anticipate applications to spinor theory [1] and anyonic exchange symmetries in physical systems, where phase is accumulated through compositions of noncommuting exchange rotors. A promising direction for future work is the development of transform and operator theories compatible with rotor-valued exchange, where orientation transport may be incorporated directly into the algebraic kernel itself, potentially yielding generalized convolutional and harmonic frameworks beyond the binary orientation of classical Clifford geometry.

REFERENCES

- [1] Élie Cartan. Les groupes bilinéaires et les systèmes de nombres complexes. *Annales de la Faculté des Sciences de Toulouse*, 12(1):B65–B99, 1898.
- [2] William Kingdon Clifford. Applications of Grassmann’s extensive algebra. *American Journal of Mathematics*, 1(4):350–358, 1878.
- [3] Hermann Grassmann. Der ort der hamilton’schen quaternionen in der ausdehnungslehre. *Mathematische Annalen*, 12:375–386, 1877.
- [4] David Hestenes and Garret Sobczyk. *Clifford Algebra to Geometric Calculus: A Unified Language for Mathematics and Physics*. D. Reidel Publishing Company, Dordrecht, 1984.
- [5] A. O. Morris. On a generalized Clifford algebra. *The Quarterly Journal of Mathematics*, 18(1):7–12, 1967.
- [6] Julian Schwinger. Unitary operator bases. *Proceedings of the National Academy of Sciences of the USA*, 46(4):570–579, 1960.
- [7] Hermann Weyl. *The Theory of Groups and Quantum Mechanics*. Methuen & Co. Ltd., London, 1931.